数理解析研究所講究録
714 巻 1990 年 160-167

SOME PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

HITOSHI SAITOH (群馬高専・斎藤 斉)

1. Introduction.

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (\, p,n \in N = \{1,2,3,\cdots\}\,) \tag{1.1}$$

which are analytic in the unit disk $U = \{ z : |z| < 1\}$.

Further, we define a function $F_\lambda(z)$ by

$$F_\lambda(z) = (1-\lambda) f(z) + \lambda z f'(z) \tag{1.2}$$

for $\lambda \geqq 0$ and $f(z) \in A(p,n)$. A function $f(z)$ belonging to $A(p,1)=A(p)$ is said to be in the class $P(p,\alpha)$ if and only if it satisfies

$$\mathrm{Re}\{ f^{(p)}(z)\} > \alpha \tag{1.3}$$

for some α ($0 \leqq \alpha < p!$) and for all $z \in U$.

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n . \tag{1.4}$$

Let the function $f(z)$ and $g(z)$ be analytic in U. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that

$$f(z) = g(w(z))$$

for $z \in U$. We denote the subordination by

$$f(z) \prec g(z) . \tag{1.5}$$

2. Inequalities for functions in the class A(p,n).

We begin with the statement of the following lemma due to Miller [1].

Lemma 1. Let $\phi(u,v)$ be a complex valued function such that

$$\phi : \mathbb{D} \to \mathbb{C}, \quad \mathbb{D} \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane }),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in $\mathbb{D}$,

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\{\phi(1,0)\} > 0$,

(iii) for all $(iu_2,v_1) \in \mathbb{D}$ such that $v_1 \leqq -n(1+u_2^2)/2$, $\mathrm{Re}\{\phi(iu_2,v_1)\} \leqq 0$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ be regular in the unit disk U such that $(p(z),zp'(z)) \in \mathbb{D}$ for all $z \in U$. If

$$\mathrm{Re}\,\{\phi(p(z),zp'(z))\} > 0 \qquad (z \in U),$$

then $\mathrm{Re}\,\{p(z)\} > 0 \quad (z \in U)$.

Applying the above lemma, we prove

Theorem 1. Let a function $f(z)$ defined by (1.1) be in the class A(p,n). If

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \qquad \left(0 \leqq \alpha < \frac{p!}{(p-j)!}\,;\ z \in U\right),$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{1}{(p-j+1)!}\,\frac{(p-j+1)!\,2\alpha + np!}{\{2(p-j+1) + n\}} \qquad (z \in U),$$

where $1 \leqq j \leqq p$.

Proof. We define the function $p(z)$ by

$$\frac{(p-j+1)!}{p!}\,\frac{f^{(j-1)}(z)}{z^{p-j+1}} = \beta + (1-\beta)p(z) \tag{2.1}$$

with $\beta = \dfrac{(p-j+1)!\,2\alpha + np!}{p!\{2(p-j+1) + n\}}$. Then $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is regular in U. Differentiating both sides in (2.1), we obtain

(2.2) $$\frac{(p-j+1)!}{p!} f^{(j)}(z) = (p-j+1)\beta z^{p-j} + (p-j+1)(1-\beta)z^{p-j}\, p(z) + (1-\beta)z^{p-j+1}\, p'(z)$$

and, by using (2.1) and (2.2), we have

(2.3) $$(p-j+1)!\left\{\frac{f^{(j)}(z)}{z^{p-j}} - \alpha\right\} = p!(p-j+1)\beta - (p-j+1)!\alpha + p!(p-j+1)(1-\beta)p(z) + p!(1-\beta)zp'(z).$$

Hence, in view of $\mathrm{Re}\,\{f^{(j)}(z)/z^{p-j}\} > \alpha$, we have

(2.4) $$\mathrm{Re}\,\{\phi(p(z), zp'(z))\} > 0,$$

where $\phi(u,v)$ is defined by

(2.5) $$\phi(u,v) = p!(p-j+1)\beta - (p-j+1)!\alpha + p!(p-j+1)(1-\beta)u + p!(1-\beta)v$$

with $u = u_1 + iu_2$, $v = v_1 + iv_2$. Then we see that

(i) $\phi(u,v)$ is continuous in $\mathbb{D} = \mathbb{C}\times\mathbb{C}$,

(ii) $(1,0)\varepsilon\mathbb{D}$ and $\mathrm{Re}\{\phi(1,0)\} = (p-j+1)!\{p!/(p-j)! - \alpha\} > 0$,

(iii) for all $(iu_2, v_1)\varepsilon\mathbb{D}$ such that $v_1 \leqq -n(1+u_2^2)/2$,

$$\mathrm{Re}\{\phi(iu_2,v_1)\} = p!(p-j+1)\beta - (p-j+1)!\alpha + p!(1-\beta)v_1$$
$$\leqq p!(p-j+1)\beta - (p-j+1)!\alpha - \frac{np!(1-\beta)(1+u_2^2)}{2} \leqq 0$$

for $\beta = \dfrac{(p-j+1)!2\alpha + np!}{p!\{2(p-j+1) + n\}} < 1$. Consequently, $\phi(u,v)$ satisfies the conditions in lemma 1. Therefore, we have $\mathrm{Re}\,\{p(z)\} > 0$ ($z\varepsilon U$), that is,

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{p!}{(p-j+1)!}\beta = \frac{1}{(p-j+1)!}\frac{(p-j+1)!2\alpha + np!}{\{2(p-j+1) + n\}}$$

which completes the proof of Theorem 1.

Taking $n = 1$ in Theorem 1, we have

Corollary 1. Let $f(z)\varepsilon A(p)=A(p,1)$ and suppose

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \qquad \left(0 \leqq \alpha < \frac{p!}{(p-j)!} \;;\; z\varepsilon U \right).$$

Then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{1}{(p-j+1)!}\,\frac{(p-j+1)!2\alpha + p!}{2(p-j)+3} \qquad (z\varepsilon U),$$

where $1 \leqq j \leqq p$.

Corollary 1 is the result by Saitoh [5].

Next, we prove

Theorem 2. Let a function $F_\lambda(z)$ defined by (1.2) for $\lambda \geqq 0$ and $f(z)\varepsilon A(p,n)$. If

$$\mathrm{Re}\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\} > \alpha \qquad \left(0 \leqq \alpha < \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\ ;\ z\varepsilon U\right),$$

then

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{(p-j)!2\alpha + np!\lambda}{(p-j)!\{2+(2p+n-2)\lambda\}} \qquad (z\varepsilon U),$$

where $0 \leqq j \leqq p$.

Proof. By the differentiation of $F_\lambda(z)$, we obtain

$$F_\lambda^{(j)}(z) = (1-\lambda+\lambda j)f^{(j)}(z) + \lambda z f^{(j+1)}(z). \tag{2.6}$$

We define the function $p(z)$ by

$$\frac{(p-j)!}{p!}\,\frac{f^{(j)}(z)}{z^{p-j}} = \beta + (1-\beta)p(z) \tag{2.7}$$

with $\beta = \dfrac{(p-j)!2\alpha + np!\lambda}{p!\{2+(2p+n-2)\}}$ $(0 \leqq \beta < 1)$. Then $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots$ is regular in U. Making the differentiation in (2.7), we have

$$\frac{zf^{(j+1)}(z)}{z^{p-j}} - \frac{p!}{(p-j+1)!}\{\beta + (1-\beta)p(z)\} = \frac{p!}{(p-j)!}(1-\beta)zp'(z). \tag{2.8}$$

By using (2.6), (2.7) and (2.8), we obtain

$$\frac{F_\lambda^{(j)}(z)}{z^{p-j}} - \alpha = \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!}p(z) + \frac{p!\lambda(1-\beta)}{(p-j)!}zp'(z). \tag{2.9}$$

Hence, in view of $\mathrm{Re}\,\{F_\lambda^{(j)}(z)/z^{p-j}\} > \alpha$, we have

$$\mathrm{Re}\,\{\phi(p(z), zp'(z))\} > 0, \tag{2.10}$$

where $\phi(u,v)$ is defined by

$$\phi(u,v) = \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!}u + \frac{p!\lambda(1-\beta)}{(p-j)!}v \tag{2.11}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then we see that

(i) $\phi(u,v)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$,

(ii) $(1,0)\varepsilon\mathbb{D}$ and $\mathrm{Re}\{\phi(1,0)\} = \frac{p!(1-\lambda+p\lambda)}{(p-j)!} - \alpha > 0$,

(iii) for all $(iu_2, v_1)\varepsilon\mathbb{D}$ such that $v_1 \leqq -n(1+u_2^2)/2$

$$\begin{aligned}\mathrm{Re}\{\phi(iu_2, v_1)\} &= \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha + \frac{p!\lambda(1-\beta)}{(p-j)!}v_1 \\ &\leqq \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\beta - \alpha - \frac{np!\lambda(1-\beta)(1+u_2^2)}{2(p-j)!} \leqq 0\end{aligned}$$

for $\beta = \frac{(p-j)!2\alpha + np!\lambda}{p!\{2+(2p+n-2)\lambda\}}$. Consequently, $\phi(u,v)$ satisfies the conditions in lemma 1. Therefore, we have

$$\mathrm{Re}\,\{p(z)\} > 0 \qquad (z\varepsilon U), \text{ that is,}$$

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{p!}{(p-j)!}\beta = \frac{(p-j)!2\alpha + np!\lambda}{(p-j)!\{2+(2p+n-2)\lambda\}}$$

which completes the assertion of Theorem 2.

Making $n = 1$ in Theorem 2, we have

Corollary 2. Let a function $F_\lambda(z)$ defined by (1.2) for $\lambda \geqq 0$ and $f(z)\varepsilon A(p)=A(p,1)$. If

$$\mathrm{Re}\left\{\frac{F_\lambda{}^{(j)}(z)}{z^{p-j}}\right\} > \alpha \qquad \left(0 \leqq \alpha < \frac{p!(1-\lambda+p\lambda)}{(p-j)!}\,;\ z\varepsilon U\right),$$

then

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{(p-j)!2\alpha + p!\lambda}{(p-j)!(2-\lambda+2p\lambda)} \qquad (z\varepsilon U),$$

where $0 \leqq j \leqq p$.

Corollary 2 is the result by Saitoh [5].

3. Some properties of the class $P(p,\alpha)$.

For giving some results in this part, we need the following lemma of Ruscheweyh and Sheil-Small [4].

Lemma 2. Let $F(z)$ and $G(z)$ be convex in U and

$f(z) \prec F(z)$. Then

$f * G(z) \prec F * G(z)$.

With the aid of above Lemma 2, we prove

Theorem 3. Let a function $f(z)$ defined by (1.3) be in the class $P(p,\alpha)$. Then we have

$$\frac{f^{(p-1)}(z)}{z} \prec 2\alpha - p! - \frac{2(p!-\alpha)}{z}\log(1-z) .$$

Proof. Define the function $G^{(p-1)}(z)$ by

$$G^{(p)}(z) = \frac{p! + (p!-2\alpha)z}{1-z}$$

and $G^{(p-1)}(0) = 0$. Then it follows that

$$\frac{G^{(p-1)}(z)}{z} = 2\alpha - p! - \frac{2(p!-\alpha)}{z}\log(1-z) .$$

Noting $f(z)\ P(p,\alpha)$, we see that

$$f^{(p)}(z) \prec G^{(p)}(z) .$$

Defining the function $k(z)$ by

$$k(z) = -\frac{\log(1-z)}{z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n \quad , \tag{3.1}$$

we have

$$\frac{f^{(p-1)}(z)}{z} = k * f^{(p)}(z) \text{ and } \frac{G^{(p-1)}(z)}{z} = k * G^{(p)}(z) .$$

Further, $k(z)$ is convex and univalent in U, and $G^{(p)}(z)$ is also convex and univalent in U. Therefore, using Lemma 2, we have

$k * f^{(p)}(z) \prec k * G^{(p)}(z)$, that is,

$$\frac{f^{(p-1)}(z)}{z} \prec \frac{G^{(p-1)}(z)}{z} = 2\alpha - p! - \frac{2(p!-\alpha)}{z}\log(1-z) .$$

Thus we completes the proof of Theorem 3.

Next, we prove

Corollary 3. Let $f(z)\varepsilon P(p,\alpha)$, Then we have

$$\mathrm{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 2\alpha - p! + 2(p!-\alpha)\log 2 .$$

Proof. Since the function $k(z)$ defined by (3.1) is convex and univalent in U, the function $G_1(z)$ given by

$$G_1(z) = 2\alpha - p! - \frac{2(p!-\alpha)}{z}\log(1-z)$$

is also convex and univalent in U. Therefore, by the principle of the subordination, we have

$$\mathrm{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > \inf_{|z|<1} \mathrm{Re}\,\{G_1(z)\} .$$

We note that $G_1(U)$ is symmetric with respect to the real axis because all coefficients of $G_1(z)$ are real. Noting that $G_1(U)$ is convex, we obtain

$$\inf_{|z|<1} \mathrm{Re}\,\{G_1(z)\} = \inf_{-1<x<1} G_1(x) = 2\alpha - p! + 2(p!-\alpha)\log 2,$$

which proves the assertion of Corollary 3.

Putting $p = 1$ in Theorem 3 and Corollary 3, we have the following corollaries which were proved by Owa, Ma and Liu [2].

Corollary 4. Let $f(z)\varepsilon A=A(1)$. If the function $f(z)$ is in the class $P(1,\alpha)=P(\alpha)$, then

$$\frac{f(z)}{z} \prec 2\alpha - 1 - \frac{2(1-\alpha)}{z}\log(1-z) .$$

Corollary 5. If $f(z)\varepsilon P(\alpha)$, then

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > 2\alpha - 1 + 2(1-\alpha)\log 2 .$$

References

[1] S.S. Miller : Differential inequalities and Carathéodory functions. Bull. Amer. Math. Soc., 81, 79-81(1975).

[2] S. Owa, W. Ma and L. Liu : On a class of analytic functirns satisfying Re{f'(z)}>α. Bull. Korean Math. Soc., 25, 211-214(1988).

[3] S. Owa and M. Nunokawa : Properties of certain analytic functions. Math. Japon., 33, 577-582(1988).

[4] S. Rucheweyh and T. Sheil-Small : Hadamard products of schlicht functions and the Pólya-Schoenberg Conjecture. Comment. Math. Helv., 48, 119-135 (1973).

[5] H. Saitoh : Properties of certain analytic functions. Proc. Japan Acad., 65, Ser. A, 131-134(1989).